

SPECTRAL CHARACTERIZATION OF THE QUADRATIC VARIATION OF MIXED BROWNIAN-FRACTIONAL BROWNIAN MOTION

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ABSTRACT. Dzharidze and Spreij [5] showed that the quadratic variation of a semimartingale can be approximated using a randomized periodogram. We show that the same approximation is valid for a special class of continuous stochastic processes. This class contains both semimartingales and non-semimartingales. The motivation comes partially from the recent work by Bender et al. [2], where it is shown that the quadratic variation of the log-returns determines the hedging strategy.

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1. INTRODUCTION

Spectral characterization of the bracket. It is well-known that for a semimartingale X , the bracket $[X, X]$ can be identified with

$$[X, X]_t = \mathbb{P}\text{-}\lim_{|\pi| \rightarrow 0} \sum_{t_k \in \pi} (X_{t_k} - X_{t_{k-1}})^2,$$

where $\pi = \{t_k : 0 = t_0 < t_1 < \dots < t_n = t\}$ is a partition of the interval $[0, t]$, $|\pi| = \max \{t_k - t_{k-1} : t_k \in \pi\}$, and $\mathbb{P}\text{-}\lim$ means convergence in probability. Statistically speaking, the sums of squared increments (*realized quadratic variation*) is a consistent estimator for the bracket as the volume of observations tends to infinity. Barndorff-Nielsen and Shephard [1] studied precision of the realized quadratic variation estimator for a special class of continuous semimartingales. They showed that sometimes the realized quadratic variation estimator can be rather noisy estimator. So one should seek for new estimators of the quadratic variation.

Dzharidze and Spreij [5] suggested another characterization of the bracket $[X, X]$. Let \mathbb{F}^X be the filtration of X and τ be a finite stopping time. For

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$\lambda \in \mathbb{R}$, define the *periodogram* $I_\tau(X; \lambda)$ of X at τ by

$$(1.1) \quad \begin{aligned} I_\tau(X; \lambda) &:= \left| \int_0^\tau e^{i\lambda s} dX_s \right|^2 \\ &= 2 \operatorname{Re} \int_0^\tau \int_0^t e^{i\lambda(t-s)} dX_s dX_t + [X, X]_\tau \quad (\text{by Ito formula}). \end{aligned}$$

Given $L > 0$ and ξ be a symmetric random variable with a density g_ξ , real characteristic function φ_ξ , and independent of the filtration \mathbb{F}^X . Define the randomized periodogram by

$$(1.2) \quad \mathbb{E}_\xi I_\tau(X; L\xi) = \int_{\mathbb{R}} I_\tau(X; Lx) g_\xi(x) dx.$$

If the characteristic function φ_ξ is of bounded variation, then Dzhangaridze and Spreij have shown that we have the following characterization of the bracket as $L \rightarrow \infty$

$$\mathbb{E}_\xi I_\tau(X; L\xi) \xrightarrow{\mathbb{P}} [X, X]_\tau.$$

Robust Black & Scholes pricing by hedging. Next we give another motivation for our estimation problem. Bender et al. [2] consider a class of pricing models, where the continuous stock price S has the following quadratic variation as a functional of the observed path S :

$$(1.3) \quad d[S, S]_t = \sigma^2(S_t) dt.$$

Here $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function of linear growth. A typical example of this kind of stock price models is the classical Black & Scholes model with constant volatility σ , where the stock price \tilde{S} is given by

$$\tilde{S}_t = s_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t},$$

where W is a standard Brownian motion. We have

$$d[\tilde{S}, \tilde{S}]_t = \sigma^2 \tilde{S}_t^2 dt,$$

and the bracket $[\tilde{S}, \tilde{S}]$ has the form of (1.3). On the other hand, let X be a continuous process with quadratic variation $[X, X]_t = \sigma^2 t$; take for example $X_t = \sigma W_t + \eta B_t^H$, B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, independent of W , and η is a constant. Then, for the process $S_t = s_0 e^{X_t - \frac{1}{2}\sigma^2 t}$, we have that

$$d[S, S]_t = \sigma^2 S_t^2 dt,$$

and again the bracket has the functional form of (1.3). Here we have examples, where the quadratic variation of the driving process X determines the structure of the quadratic variation of the stock price. Moreover, if this is the case, then Bender et al. have shown that within a fixed model class, determined by the relation (1.3) the hedging of options has the same functional form as in the classical Black & Scholes pricing model. The options, which can be hedged, includes European options, path dependent options like look-back options, and Asian options.

The results. We show that the result of Dzhaparidze and Spreij holds for the mixed Brownian-fractional Brownian motion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and fix $T > 0$. Throughout the paper, we assume that $W = \{W_t\}_{t \in [0, T]}$ is a standard Brownian motion and $B^H = \{B_t^H\}_{t \in [0, T]}$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, independent of the Brownian motion W . Define the mixed Brownian-fractional Brownian motion X_t by

$$X_t = W_t + B_t^H \quad t \in [0, T].$$

It is known that (see [3]) the process X is a $(\mathbb{F}^X, \mathbb{P})$ semimartingale, if $H \in (\frac{3}{4}, 1)$, and for $H \in (\frac{1}{2}, \frac{3}{4}]$, X is not a semimartingale with respect to its own filtration \mathbb{F}^X . Moreover in both cases we have

$$(1.4) \quad [X, X]_t = [X]_t = \mathbb{P}\text{-}\lim_{|\pi| \rightarrow 0} \sum_{t_i \leq t} (X_{t_i} - X_{t_{i-1}})^2 = t.$$

If the partitions in (1.4) are nested, then the convergence can be strengthened to almost sure convergence. Hereafter, we always assume that the sequence of partitions are nested. For $\lambda \in \mathbb{R}$, define the complex-valued stochastic process Y by

$$Y_t = \int_0^t e^{i\lambda s} dX_s \quad t \in [0, T],$$

where the stochastic integral is understood in a path-wise way, and it is defined by integration by parts formula (see [12]):

$$\int_0^t e^{i\lambda s} dX_s = e^{i\lambda t} X_t - i\lambda \int_0^t X_s e^{i\lambda s} ds.$$

Therefore, $Y = \{Y_t\}_{t \in [0, T]}$ is a process with continuous sample paths. Moreover, it is straightforward to check that for $t \in [0, T]$, we have that

$$\begin{aligned} [Y, Y]_t &= [Y]_t := \text{a.s.-}\lim_{|\pi| \rightarrow 0} \sum_{t_i \leq t} \left((Y_{t_k} - Y_{t_{k-1}}) (\overline{Y_{t_k} - Y_{t_{k-1}}}) \right) \\ &= [\mathbf{Re} Y]_t + [\mathbf{Im} Y]_t = [X]_t = t, \end{aligned}$$

where \overline{Y}_t is complex conjugate of Y_t ([12, p.84]). Given $\lambda \in \mathbb{R}$, define the periodogram of X at T as (1.1), i.e.

$$\begin{aligned} I_T(X; \lambda) &= \left| \int_0^T e^{i\lambda t} dX_t \right|^2 \\ &= \left| e^{i\lambda T} X_T - i\lambda \int_0^T X_t e^{i\lambda t} dt \right|^2 \\ &= X_T^2 + X_T \int_0^T i\lambda (e^{i\lambda(T-t)} - e^{-i\lambda(T-t)}) X_t dt + \lambda^2 \left| \int_0^T e^{i\lambda t} X_t dt \right|^2. \end{aligned}$$

Assume $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be an another probability space and identify the σ -algebra \mathcal{F} by $\mathcal{F} \otimes \{\phi, \tilde{\Omega}\}$ on the product space $(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, \mathbb{P} \otimes \tilde{\mathbb{P}})$. Let $\xi : \tilde{\Omega} \rightarrow \mathbb{R}$ be a real symmetric random variable with density g_ξ , and independent of

the filtration \mathbb{F}^X . Define for any positive real number L the randomized periodogram by

$$(1.5) \quad \mathbb{E}_\xi I_T(X; L\xi) := \int_{\mathbb{R}} I_T(X; Lx) g_\xi(x) dx.$$

Our main result is the following.

Theorem 1.1. *Assume that X is a mixed Brownian-fractional Brownian motion, $\mathbb{E}_\xi I_T(X; L\xi)$ is the randomized periodogram given by (1.5) and*

$$\mathbb{E}\xi^2 < \infty.$$

Then as $L \rightarrow \infty$ we have

$$\mathbb{E}_\xi I_T(X; L\xi) \xrightarrow{\mathbb{P}} [X, X]_T.$$

Remark 1.1. *To compare our result to the results of Dzhaparidze and Spreij:*

- *They take any finite stopping time τ , whereas we must assume a constant stopping time T .*
- *They assume that the characteristic function φ_ξ is of bounded variation, whereas instead we assume that ξ is a square integrable random variable.*

Note that under our assumption of deterministic stopping time $\tau = T$, when X is a Gaussian martingale, they can drop the condition of the bounded variation on $[0, \infty)$ of the characteristic function φ_ξ of the random variable ξ (see [5, Remark, p.170]).

Next we give some auxiliary material and then finish the proof.

2. AUXILIARY RESULTS

2.1. Path-wise Ito formula. Föllmer [6] obtained a path-wise calculus for continuous functions with finite quadratic variation. The next theorem is essentially due to Föllmer. For a nice exposition, and its use in finance, see Sondermann [14].

Theorem 2.1. [14] *Let $X : [0, T] \rightarrow \mathbb{R}$ be a continuous process with continuous quadratic variation $[X, X]_t$ and $F \in C^2(\mathbb{R})$. Then for any $t \in [0, T]$, the limit of the Riemann-Stieltjes sums*

$$\lim_{|\pi| \rightarrow 0} \sum_{t_i \leq t} F_x(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) := \int_0^t F_x(X_s) dX_s,$$

exists almost surely. Moreover, we have

$$(2.1) \quad F(X_t) = F(X_0) + \int_0^t F_x(X_s) dX_s + \frac{1}{2} \int_0^t F_{xx}(X_s) d[X, X]_s.$$

Lemma 2.1. *For the mixed Brownian-fractional Brownian motion X we have*

$$(2.2) \quad I_T(X; \lambda) = [X]_T + 2 \operatorname{Re} \int_0^T \int_0^t e^{i\lambda(t-s)} dX_s dX_t$$

where the iterated stochastic integral in the right hand side is understood in path-wise way, i.e. as the limit of the Riemann-Stieltjes sums.

Proof: We apply Ito type formula (2.1) to the real part $\mathbf{Re} Y$ and the imaginary part $\mathbf{Im} Y$ of the process $Y = \{Y_t\}_{t \in [0, T]}$ with the function $F(x) = x^2$. We obtain

$$F(\mathbf{Re} Y_T) = \int_0^T \left(2 \int_0^t \mathbf{Re} e^{i\lambda s} dX_s \right) \mathbf{Re} e^{i\lambda t} dX_t + [\mathbf{Re} Y]_T.$$

Similarly, we have

$$F(\mathbf{Im} Y_T) = \int_0^T \left(2 \int_0^t \mathbf{Re} -ie^{i\lambda s} dX_s \right) \mathbf{Re} -ie^{i\lambda t} dX_t + [\mathbf{Im} Y]_T.$$

Summing the left and right hand sides of the identities, we get

$$\begin{aligned} \left| \int_0^T e^{i\lambda t} dX_t \right|^2 &= \left(\int_0^T \mathbf{Re} e^{i\lambda t} dX_t \right)^2 + \left(\int_0^T \mathbf{Re} -ie^{i\lambda t} dX_t \right)^2 \\ &= [X]_T + 2 \int_0^T \int_0^t \mathbf{Re} e^{i\lambda s} \mathbf{Re} e^{i\lambda t} dX_s dX_t \\ &\quad + 2 \int_0^T \int_0^t -\mathbf{Re} ie^{i\lambda s} -\mathbf{Re} ie^{i\lambda t} dX_s dX_t \\ &= [X]_T + 2 \mathbf{Re} \int_0^T \int_0^t e^{i\lambda(t-s)} dX_s dX_t. \end{aligned}$$

Note that $\mathbf{Re} e^{i\lambda(t-s)} = \mathbf{Re} e^{i\lambda s} \mathbf{Re} e^{i\lambda t} + \mathbf{Re} -ie^{i\lambda s} \mathbf{Re} -ie^{i\lambda t}$.

2.2. Path-wise stochastic integration in fractional Besov-type spaces.

A stochastic process X is a semimartingale if and only if one has a version of the Lebesgue dominated convergence theorem (see [12]). Fractional Brownian motion is not a semimartingale, and hence the stochastic integral with respect to fractional Brownian motion B^H must be defined. Using the smoothness of the sample paths of the fractional Brownian motion B^H , when $H \in (\frac{1}{2}, 1)$, one can define the so-called *generalized Lebesgue-Stieltjes integral*. For more information, see [10], [16] and [9].

Definition 2.1. [10] Fix $0 < \alpha < \frac{1}{2}$.

(i) For $f : [0, T] \rightarrow \mathbb{R}$, define

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(t)|}{t^\alpha} dt + \int_0^T \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds dt,$$

and

$$W_0^{\alpha,1}[0, T] = \{f : [0, T] \rightarrow \mathbb{R} ; \|f\|_{\alpha,1} < \infty\}.$$

(ii) Also, for $f : [0, T] \rightarrow \mathbb{R}$, define

$$\|f\|_{1-\alpha,\infty,T} := \sup_{0 < s < t < T} \left(\frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|f(y) - f(s)|}{(y-s)^{2-\alpha}} dy \right),$$

and

$$W_T^{1-\alpha,\infty}[0, T] := \{f : [0, T] \rightarrow \mathbb{R} ; \|f\|_{1-\alpha,\infty,T} < \infty\}.$$

Denote by $C^\lambda[0, T]$ the space of λ -Hölder continuous functions on the interval $[0, T]$. Then $\forall \epsilon > 0$, we have the inclusions

$$\begin{aligned} C^{1-\alpha+\epsilon}[0, T] &\subseteq W_T^{1-\alpha, \infty}[0, T] \subseteq C^{1-\alpha}[0, T] \\ C^{\alpha+\epsilon}[0, T] &\subseteq W_0^{\alpha, 1}[0, T]. \end{aligned}$$

Recall that almost surely sample paths of B^H for any $0 < \gamma < H$, belong to $C^\gamma[0, T]$. This follows from the Kolmogorov continuity theorem. Hence the sample paths of B^H belong to $W_T^{\alpha, \infty}[0, T]$ for any $0 < \alpha < H$.

In the following D_{t-}^α (resp. D_{0+}^α) stand for right-sided (resp. left-sided) fractional derivatives ([13]). For $g \in W_T^{1-\alpha, \infty}[0, T]$, define

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0 < s < t < T} |(D_{t-}^{1-\alpha} g_{t-})(s)| \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha, \infty, T}.$$

Definition 2.2. [10] Fix $0 < \alpha < \frac{1}{2}$. Let $f \in W_0^{\alpha, 1}[0, T]$ and $g \in W_T^{1-\alpha, \infty}[0, T]$. Then the Lebesgue integral

$$\int_0^T D_{0+}^\alpha f_{0+}(t) D_{T-}^{1-\alpha} g_{T-}(t) dt$$

exists, and we can define the generalized Lebesgue-Stieltjes integral by

$$\int_0^T f_t dg_t := \int_0^T D_{0+}^\alpha f_{0+}(t) D_{T-}^{1-\alpha} g_{T-}(t) dt,$$

where $f_{0+}(t) = f(t) - f(0^+)$ and $g_{T-}(t) = g(T^-) - g(t)$.

Remark 2.1. [9], [16] The definition of the generalized Lebesgue-Stieltjes integral does not depend on the choice of α .

Remark 2.2. [16] If f and g are Hölder continuous of orders α and β with $\alpha + \beta > 1$, then the generalized Lebesgue-Stieltjes integral exists and coincides with the Riemann-Stieltjes integral. This fact is based on the integration theory developed by Young [15].

Since fractional Brownian motion is not a semimartingale, the next theorem and corollary can be used instead of the Lebesgue dominated convergence theorem for fractional Brownian motion.

Theorem 2.2. [10] Let $g \in W_T^{1-\alpha, \infty}[0, T]$ and $f \in W_0^{\alpha, 1}[0, T]$. Then we have the estimate

$$\left| \int_0^T f_t dg_t \right| \leq \Lambda_\alpha(g) C_{\alpha, T} \|f\|_{\alpha, 1}$$

for some constant $C = C_{\alpha, T}$.

Corollary 2.1. Assume $f, f^n \in W_0^{\alpha, 1}[0, T]$, and $\|f^n - f\|_{\alpha, 1} \rightarrow 0$ as $n \rightarrow \infty$ for some $\alpha \in (1 - H, \frac{1}{2})$. Then as $n \rightarrow \infty$

$$\int_0^T f_t^n dB_t^H \rightarrow \int_0^T f_t dB_t^H, \quad a.s.$$

Next we use this machinery to prove a stochastic Fubini type result that is cornerstone of the proof of our main result.

Proposition 2.1. *Assume that $X = W + B^H$ is a mixed Brownian-fractional Brownian motion, and ξ is a square integrable random variable with a density g_ξ , and independent of the filtration \mathbb{F}^X . Then the iterated integrals*

$$I_1 = \int_{\mathbb{R}} \left(\int_0^T \phi_W(t, x) dB_t^H \right) g_\xi(x) dx$$

$$I_2 = \int_0^T \left(\int_{\mathbb{R}} \phi_W(t, x) g_\xi(x) dx \right) dB_t^H$$

exist, and moreover $I_1 = I_2$ almost surely, where $\phi_W(t, x) = \int_0^t e^{ix(t-s)} dW_s$ and the stochastic integrals in I_1 and I_2 are understood in path-wise way as the limit of Riemann-Stieltjes sums.

Proof: We split the proof into four steps.

Step 1: The existence of I_1 . Using integration by parts formula and simple manipulations, we see that the sample paths of the complex-valued stochastic process $\{\phi_W(t, x)\}_{t \in [0, T]}$ parametrized by $x \in \mathbb{R}$, are Hölder continuous of any order less than half almost surely. Moreover, for any $\alpha \in (0, \frac{1}{2})$, $x \in \mathbb{R}$, and $s, t \in [0, T]$, we have

$$(2.3) \quad \begin{aligned} |\phi_W(t, x) - \phi_W(s, x)| &\leq C(\omega, T)(1 + |x| + |x|^2)|t - s|^\alpha \text{ and} \\ |\phi_W(t, x)| &\leq C(\omega, T)(1 + T|x|), \end{aligned}$$

where $C(\omega, T)$ is an almost surely finite and positive random variable that may be different from line to line. Hence the interior stochastic integral in I_1 can be defined as limit of Riemann-Stieltjes sums almost surely by using the Young integration theory (see Remark 2.2). Note that by (2.3), the function

$$x \mapsto \int_0^T \phi_W(t, x) dB_t^H$$

is integrable with respect to the measure $g_\xi(x)dx$ almost surely.

Step 2: The existence of I_2 . Using (2.3), we see that for any $\alpha \in (0, \frac{1}{2})$,

$$\left| \int_{\mathbb{R}} \phi_W(t, x) g_\xi(x) dx - \int_{\mathbb{R}} \phi_W(s, x) g_\xi(x) dx \right| \leq C(\omega, T)(1 + 2\mathbb{E}\xi^2)|t - s|^\alpha.$$

Therefore, the stochastic integral I_2 can be defined as limit of the Riemann-Stieltjes sums almost surely.

Step 3. Define for any $N \in \mathbb{N}$,

$$I_1^N = \int_{-N}^N \left(\int_0^T \phi_W(t, x) dB_t^H \right) g_\xi(x) dx$$

$$I_2^N = \int_0^T \left(\int_{-N}^N \phi_W(t, x) g_\xi(x) dx \right) dB_t^H.$$

Clearly I_1^N converges to I_1 almost surely as N tends to infinity. We aim to show that $I_1^N = I_2^N$ almost surely. By definition of the Riemann integral,

there exists a sequence of partitions $\{\pi_n^N\}_{n=1}^\infty$ of the interval $[-N, N]$, such that $|\pi_n^N| \rightarrow 0$ as $n \rightarrow \infty$ and

$$I_1^N = \lim_{n \rightarrow \infty} \sum_{x_i^n \in \pi_n^N} \left(\int_0^T \phi_W(t, x_{i-1}^n) dB_t^H \right) g_\xi(x_{i-1}^n) \Delta x_i^n,$$

and

$$\lim_{n \rightarrow \infty} \sum_{x_i^n \in \pi_n^N} \phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) \Delta x_i^n = \int_N^N \phi_W(t, x) g_\xi(x) dx$$

hold. Assume that $\pi_n^N = \{-N = x_0^n < x_1^n < \dots < x_{k_n}^n = N\}$. For each $x_{i-1}^n \in \pi_n^N$, $1 \leq i \leq k_n + 1$, we can find a sequence $\{\pi_m^{T, x_{i-1}^n}\}$ of partitions of the interval $[0, T]$, such that $|\pi_m^{T, x_{i-1}^n}| \rightarrow 0$ as $m \rightarrow \infty$ and

$$\lim_{m \rightarrow \infty} \sum_{t_j^m \in \pi_m^{T, x_{i-1}^n}} \phi_W(t_j^m, x_{i-1}^n) \Delta B_{t_j^m}^H = \int_0^T \phi_W(t, x_{i-1}^n) dB_t^H.$$

On the other hand, there is another sequence $\{\hat{\pi}_m^T\}$ of partitions of the interval $[0, T]$ such that

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{\hat{t}_j^m \in \hat{\pi}_m^T} \left(\int_{-N}^N \phi_W(\hat{t}_{j-1}^m, x) g_\xi(x) dx \right) \Delta B_{\hat{t}_j^m}^H \\ = \int_0^T \left(\int_{-N}^N \phi_W(t, x) g_\xi(x) dx \right) dB_t^H. \end{aligned}$$

Let

$$\pi_m^{T, n} = \cup_{i=1}^{k_n+1} \pi_m^{T, x_{i-1}^n} \quad \text{and} \quad \pi_m^T = \hat{\pi}_m^T \cup \pi_m^{T, n}.$$

Therefore, for any $n \in \mathbb{N}$, the partition π_m^T of the interval $[0, T]$ contains all points of the partitions $\pi_m^{T, n}$ and $\hat{\pi}_m^T$, and denote the points of π_m^T by t_k^m , $k = 0, \dots, l_m$. Then for any $x_{i-1}^n \in \pi_n^N$, we can write

$$\lim_{m \rightarrow \infty} \sum_{t_j^m \in \pi_m^T} \phi_W(t_j^m, x_{i-1}^n) \Delta B_{t_j^m}^H = \int_0^T \phi_W(t, x_{i-1}^n) dB_t^H.$$

Now for $n, m \in \mathbb{N}$, we can have the estimate

$$|I_1^N - I_2^N| \leq |I_1^N - \Delta_{n, m}| + |I_2^N - \Delta_{n, m}| := A_{n, m} + B_{n, m},$$

where

$$\Delta_{n, m} := \sum_{x_i^n \in \pi_n^N} \sum_{t_j^m \in \pi_m^T} \phi_W(t_j^m, x_{i-1}^n) \Delta B_{t_j^m}^H g_\xi(x_{i-1}^n) \Delta x_i^n.$$

For the first term $A_{n,m}$, we have

$$\begin{aligned} |A_{n,m}| &\leq \left| I_1^N - \sum_{x_i^n \in \pi_n^N} \left(\int_0^T \phi_W(t, x_{i-1}^n) dB_t^H \right) g_\xi(x_{i-1}^n) \Delta x_i^n \right| \\ &\quad + \sum_{x_i^n \in \pi_n^N} \left| \int_0^T \phi_W(t, x_{i-1}^n) dB_t^H \right. \\ &\quad \left. - \sum_{t_j^m \in \pi_m^T} \phi_W(t_{j-1}^m, x_{i-1}^n) \Delta B_{t_j^m}^H \right| g_\xi(x_{i-1}^n) \Delta x_i^n. \end{aligned}$$

For fix n and x_{i-1}^n , when m tends to infinity, we have

$$\left| \int_0^T \phi_W(t, x_{i-1}^n) dB_t^H - \sum_{t_j^m \in \pi_m^T} \phi_W(t_{j-1}^m, x_{i-1}^n) \Delta B_{t_j^m}^H \right| \rightarrow 0.$$

Therefore, $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} A_{n,m} = 0$. Similarly, for the second term $B_{n,m}$

$$\begin{aligned} |B_{n,m}| &\leq \left| I_2^N - \sum_{t_j^m \in \pi_m^T} \left(\int_{-N}^N \phi_W(t_{j-1}^m, x) g_\xi(x) dx \right) \Delta B_{t_j^m}^H \right| \\ &\quad + \left| \Delta_{n,m} - \sum_{t_j^m \in \pi_m^T} \left(\int_{-N}^N \phi_W(t_{j-1}^m, x) g_\xi(x) dx \right) \Delta B_{t_j^m}^H \right|. \end{aligned}$$

So, it is enough to show that the second term in the right hand side converges to 0 as n, m tend to infinity. Note that the second term can be written as

$$\left| \Delta_{n,m} - \sum_{t_j^m \in \pi_m^T} \left(\int_{-N}^N \phi_W(t_{j-1}^m, x) g_\xi(x) dx \right) \Delta B_{t_j^m}^H \right| = \left| \sum_{i=0}^{k_n} \int_{x_{i-1}^n}^{x_i^n} f_m(x, x_{i-1}^n) dx \right|,$$

where

$$f_m(x, x_{i-1}^n) = \sum_{t_j^m \in \pi_m^T} (\phi_W(t_{j-1}^m, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t_{j-1}^m, x) g_\xi(x)) \Delta B_{t_j^m}^H.$$

So, when m tends to infinity, we have that

$$f_m(x, x_{i-1}^n) \rightarrow \int_0^T (\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t, x) g_\xi(x)) dB_t^H.$$

Moreover, for each $1 \leq i \leq k_n$, the sequence $f_m(x, x_{i-1}^n)$ has an integrable dominant with respect to variable x . To see this, take $\theta \in (\frac{1}{2}, H)$ and

$\lambda \in (0, \frac{1}{2})$ such that $\theta + \lambda = 1 + \epsilon$. Then

$$\begin{aligned}
|f_m(x, x_{i-1}^n)| &\leq \left| \sum_{t_j^m \in \pi_m^T} (\phi_W(t_j^m, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t_j^m, x) g_\xi(x)) \Delta B_{t_j^m}^H \right. \\
&\quad \left. - \int_0^T (\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t, x) g_\xi(x)) dB_t^H \right| \\
&\quad + \left| \int_0^T (\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t, x) g_\xi(x)) dB_t^H \right| \\
&\leq C |\pi_m^T|^\epsilon \|B^H\|_{C^\theta[0,T]} \|\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t, x) g_\xi(x)\|_{C^\lambda[0,T]} \\
&\leq C |\pi_m^T|^\epsilon \|B^H\|_{C^\theta[0,T]} \left[\|\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n)\|_{C^\lambda[0,T]} \right. \\
&\quad \left. + \|\phi_W(t, x) g_\xi(x)\|_{C^\lambda[0,T]} \right].
\end{aligned}$$

By the inequalities were obtained in (2.3), we see that

$$\|\phi_W(t, x) g_\xi(x)\|_{C^\lambda[0,T]} \leq C(\omega, T)(1 + |x| + |x|^2) g_\xi(x) \in L^1[-N, N].$$

Therefore, by the Lebesgue dominated convergence theorem, we have that as m tends to infinity

$$\begin{aligned}
&\int_{x_{i-1}^n}^{x_i^n} f_m(x, x_{i-1}^n) dx \\
&\quad \rightarrow \int_{x_{i-1}^n}^{x_i^n} \left(\int_0^T (\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t, x) g_\xi(x)) dB_t^H \right) dx.
\end{aligned}$$

Therefore, as n tends to infinity, we have

$$\sum_{i=0}^{k_n} \int_{x_{i-1}^n}^{x_i^n} \left(\int_0^T (\phi_W(t, x_{i-1}^n) g_\xi(x_{i-1}^n) - \phi_W(t, x) g_\xi(x)) dB_t^H \right) dx \rightarrow 0.$$

Hence, we have shown that $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} B_{n,m} = 0$.

Remark 2.3. *The result of this step can be derived from theorem 2.6.5 of [9, p.177], with some modifications.*

Step 4: We want to show that I_2^N converges to I_2 as N tends to infinity. Clearly, the difference is

$$(2.4) \quad |I_2 - I_2^N| = \left| \int_0^T u_t^N dB_t^H \right|$$

where

$$u_t^N := \int_{[-N, N]^c} \phi_W(t, x) g_\xi(x) dx.$$

According to Corollary 2.1, it is sufficient to show that for some $\alpha \in (1 - H, \frac{1}{2})$, the sequence $u^N \in W_0^{\alpha,1}[0, T]$ and $\|u^N\|_{\alpha,1} \rightarrow 0$. Note that the sample paths of the process u^N are Hölder continuous of any order less than half almost surely. Therefore, by the Remark 2.2, the stochastic integral

appears in (2.4) coincides with the Riemann-Stieltjes integral. Now, for any $\alpha \in (1 - H, \frac{1}{2})$, using (2.3) and the assumption $\mathbb{E}\xi^2 < \infty$, we have

$$\int_0^T \frac{|u_t^N|}{t^\alpha} dt \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

by the Lebesgue dominated convergence theorem. For the second term, we take a positive real number $\beta \in (\alpha, \frac{1}{2})$. Then using (2.3), we have

$$\begin{aligned} \int_0^T \int_0^t \frac{|u_t^N - u_s^N|}{(t-s)^{\alpha+1}} ds dt &\leq C(\omega, T) \int_0^T \int_0^s \frac{1}{(t-s)^{\alpha+1-\beta}} ds dt \\ &\quad \int_{\mathbb{R}} \mathbf{1}_{[-N, N]^c} (1 + |x| + |x|^2) g_\xi(x) dx \\ &\rightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

by the Lebesgue dominated convergence theorem, since $\alpha + 1 - \beta < 1$. Hence, we have shown that $\|u^N\|_{\alpha, 1} \rightarrow 0$ as N tends to infinity.

3. PROOF OF THE MAIN RESULT

Let φ_ξ stands for the real valued characteristic function of ξ . Then the parametrized stochastic Fubini theorem for semimartingales (see [12]), Lemma (2.1) and Proposition (2.1) allow us to write the randomized periodogram of the mixed Brownian-fractional Brownian motion X as

$$\begin{aligned} \mathbb{E}_\xi I_T(X; L\xi) &= 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dX_s dX_t + [X]_T \\ &= 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dW_s dW_t + 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dW_s dB_t^H \\ &\quad + 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_s^H dW_t + 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_s^H dB_t^H + [X]_T \\ &= 2J_1 + 2J_2 + 2J_3 + 2J_4 + [X]_T. \end{aligned}$$

Next, we show that as $L \rightarrow \infty$

$$J_k \xrightarrow{\mathbb{P}} 0, \quad k = 1, 2, 3, 4,$$

using the facts that

$$|\varphi_\xi| \leq 1 \quad \text{and} \quad \varphi_\xi(L(t-s)) \rightarrow 0 \quad \text{for } s < t \quad \text{as } L \rightarrow \infty.$$

$$J_1 \xrightarrow{\mathbb{P}} 0 :$$

By Ito isometry, we have

$$\mathbb{E} J_1^2 = \int_0^T \int_0^t \varphi_\xi^2(L(t-s)) ds dt \rightarrow 0 \quad \text{as } L \rightarrow \infty.$$

$J_2 \xrightarrow{\mathbb{P}} 0$:

Since Brownian motion W and fractional Brownian motion B^H are independent, we can compute

$$\begin{aligned}
& \mathbb{E} \left(\int_0^T \int_0^t \varphi_\xi(L(t-s)) dW_s dB_t^H \right)^2 \\
&= \mathbb{E} \left(\mathbb{E} \left(\int_0^T \int_0^t \varphi_\xi(L(t-s)) dW_s dB_t^H \right)^2 \mid \mathbb{F}_T^W \right) \\
&= H(2H-1) \mathbb{E} \int_0^T \int_0^T |u-v|^{2H-2} \\
&\quad \int_0^u \varphi_\xi(L(u-s)) dW_s \int_0^v \varphi_\xi(L(v-s)) dW_s du dv \\
&= H(2H-1) \int_0^T \int_0^T |u-v|^{2H-2} \int_0^{u \wedge v} \varphi_\xi(L(u-s)) \varphi_\xi(L(v-s)) ds du dv \\
&\rightarrow 0 \quad \text{as } L \rightarrow \infty,
\end{aligned}$$

by the Lebesgue dominated convergence theorem.

$J_3 \xrightarrow{\mathbb{P}} 0$:

Similar to the case J_2 , we can compute

$$\begin{aligned}
& \mathbb{E} \left(\int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_s^H dW_t \right)^2 \\
&= \mathbb{E} \left(\mathbb{E} \left(\int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_s^H dW_t \right)^2 \mid \mathbb{F}_T^{B^H} \right) \\
&= H(2H-1) \int_0^T \int_0^t \int_0^t \varphi_\xi(L(t-u)) \varphi_\xi(L(t-v)) |u-v|^{2H-2} du dv dt \\
&\rightarrow 0 \quad \text{as } L \rightarrow \infty.
\end{aligned}$$

$J_4 \xrightarrow{\mathbb{P}} 0$:

By theorem 4.1, [4] and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
& \mathbb{E} \left(\int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_s^H dB_t^H \right)^2 \\
&= (H(2H-1))^2 \int_0^T \int_0^T \int_0^v \int_0^u \varphi_\xi(L(u-s)) \varphi_\xi(L(v-t)) \\
&\quad |t-s|^{2H-2} |u-v|^{2H-2} ds dt du dv \\
&\rightarrow 0 \quad \text{as } L \rightarrow \infty.
\end{aligned}$$

4. MORE PROPERTIES AND REMARKS

Assume that X is a mixed Brownian-fractional Brownian motion, i.e. $X_t = W_t + B_t^H$. Let $\pi = \{t_0 = 0 < t_1 < \dots < t_n = T\}$ be a partition of

the interval $[0, T]$. Then, we have the following properties of the realized quadratic variation estimator.

- Using the Ito type formula (2.1), we have a representation for the error term, denoted by e^1 , as

$$\sum_{t_k \in \pi} (X_{t_k} - X_{t_{k-1}})^2 - [X, X]_T = 2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t dX_s dX_t.$$

- Hence, for the error term e^1 of the realized quadratic variation estimator, we obtain

$$\begin{aligned} \mathbb{E}(e^1) &= \mathbb{E}\left(2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t dX_s dX_t\right) \\ &= \mathbb{E}\left(2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t dB_s^H dB_t^H\right) = \sum_{t_k \in \pi} (\Delta t_k)^{2H}. \end{aligned}$$

This implies that the realized quadratic variation is a *biased* estimator of the quadratic variation $[X, X]$.

- Moreover, its variance is given by

$$\begin{aligned} \mathbb{V}\text{ar}(e^1) &= \mathbb{V}\text{ar}\left(2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t dX_s dX_t\right) \\ &= \sum_{k=1}^n \mathbb{V}\text{ar}\left(2 \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^t dX_s dX_t\right) \\ &\quad + 2 \sum_{\substack{1 \leq i, j \leq n \\ i < j}} \mathbb{C}\text{ov}\left(2 \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^t dX_s dX_t, 2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dX_s dX_t\right) \\ &= \sum_{k=1}^n 2 \left((\Delta t_k) + (\Delta t_k)^{2H} \right)^2 + 4 \sum_{\substack{1 \leq i, j \leq n \\ i < j}} \left(\mathbb{E}(\Delta B_{t_i}^H)^2 (\Delta B_{t_j}^H)^2 \right)^2 \\ &= \sum_{k=1}^n 2 \left((\Delta t_k) + (\Delta t_k)^{2H} \right)^2 \\ &\quad + \sum_{\substack{1 \leq i, j \leq n \\ i < j}} \left((t_j - t_{i-1})^{2H} + (t_{j-1} - t_i)^{2H} - (t_j - t_i)^{2H} - (t_{j-1} - t_{i-1})^{2H} \right)^2. \end{aligned}$$

- For the special case of equidistant partition $\pi_n = \{\frac{kT}{n}; k = 0, 1, \dots, n\}$, the mean and the variance of the error term $e^1 = e_{\pi_n}^1$ take the forms

$$\begin{aligned} \mathbb{E}(e_{\pi_n}^1) &= T^{2H} n^{1-2H}, \\ \mathbb{V}\text{ar}(e_{\pi_n}^1) &= 2n \left(\left(\frac{T}{n}\right) + \left(\frac{T}{n}\right)^{2H} \right)^2 \\ &\quad + \left(\frac{T}{n}\right)^{4H} \sum_{\substack{1 \leq i, j \leq n \\ i < j}} \left((j - i - 1)^{2H} + (j - i + 1)^{2H} - 2(j - i)^{2H} \right)^2 \end{aligned}$$

Therefore, we have the asymptotic behaviors

$$\begin{aligned}\mathbb{E}(e_{\pi_n}^1) &\sim T \quad \text{as } H \downarrow \frac{1}{2}, \\ \mathbb{E}(e_{\pi_n}^1) &\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \forall H > \frac{1}{2}, \\ \mathbb{V}\text{ar}(e_{\pi_n}^1) &\sim 2n \left(\frac{T}{n}\right)^2 = 8 \frac{T^2}{n} \quad \text{as } H \downarrow \frac{1}{2}.\end{aligned}$$

Hence, we see that

$$\lim_{n \rightarrow \infty} \lim_{H \downarrow \frac{1}{2}} \mathbb{V}\text{ar}(e_{\pi_n}^1) = 0,$$

whereas for two independent Brownian motions W^1 and W^2 , with $Z_t = W_t^1 + W_t^2$ and a simple computation we have

$$\mathbb{V}\text{ar}\left(2 \int_0^T \int_0^t dZ_s dZ_t\right) = \mathbb{V}\text{ar}\left(Z_T^2 - [Z, Z]_T\right) = 8T^2.$$

For randomized periodogram, we have the following properties.

- The error term, denoted by e^2 , of the randomized periodogram takes a form as

$$\mathbb{E}_\xi I_T(X; L\xi) - [X, X]_T = 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dX_s dX_t.$$

- The mean of the error term e^2 can be computed as

$$\begin{aligned}\mathbb{E}(e^2) &= \mathbb{E}\left(2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dX_s dX_t\right) \\ &= 2H(2H-1) \int_0^T \int_0^t \varphi_\xi(L(t-s)) |t-s|^{2H-2} ds dt.\end{aligned}$$

Therefore, the randomized periodogram is also a biased estimator of the quadratic variation $[X, X]$.

Remark 4.1. *It would be interesting to know, whether the estimating based on “discretized” periodogram (or “realized periodogram”) is less noisy than the realized quadratic variation estimator.*

Remark 4.2. *It is also interesting whether one can give an unbiased estimator of the quadratic variation of mixed Brownian-fractional Brownian motion.*

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